Integration by Substitution

In this section we reverse the Chain rule of differentiation and derive a method for solving integrals called the method of substitution. Recall the chain rule of differentiation says that

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

Reversing this rule tells us that

$$\int f'(g(x))g'(x)\, dx = f(g(x)) + C$$

Example Use the chain rule to find the derivative of the composite function $f(g(x)) = (x^2+1)^2$ and identify f and g in the expression.

Write the integral below as $\int f'(g(x))g'(x)\, dx$ and evaluate it :

$$\int 4x(x^2+1)\, dx$$

The Substitution Rule says that if $g(x)$ is a differentiable function whose range is the interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\, dx = \int f(u)\, du$$

where $u = g(x)$ and $du = g'(x)\, dx$.

When applying the method, we substitute $u = g(x)$, integrate with respect to the variable u and then reverse the substitution in the resulting antiderivative.

Example Find $\int 2x\sqrt{x^2+1}\, dx$.
Here we let $g(x) = x^2+1$. We have $\int 2x\sqrt{x^2+1}\, dx = \int \sqrt{g(x)}g'(x)\, dx$.
Now we let $u = g(x) = x^2+1$, giving us that $\frac{du}{dx} = 2x$, giving us that $du = 2xdx$. Therefore, we have

$$\int 2x\sqrt{x^2+1}\, dx = \int \sqrt{u}\, du = \frac{2u^{3/2}}{3} + C.$$

We convert our answer back to an answer in terms of the variable x, to get

$$\int 2x\sqrt{x^2+1}\, dx = \frac{2u^{3/2}}{3} + C = \frac{2(x^2+1)^{3/2}}{3} + C.$$

You should check that this the general antiderivative for $2x\sqrt{x^2+1}$, by differentiating it using the chain rule.

Sometimes your substitution may result in an integral of the form $\int f(u)c\,du$ for some constant c, which is not a problem.

Example Find the following:

$$\int x^3\sqrt{x^4+1}\,dx, \qquad \int \sin^3 x \cos x\,dx, \qquad \int x\sin(x^2+3)\,dx$$

Sometimes the appropriate substitution is non-obvious and you may have to work a little harder to put the resulting integral in the form $\int f(u)du$:

Example Find the following :

$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx$$

The Definite Integral

The Substitution Rule For Definite Integrals If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Proof If F is an antiderivative for f, we have

$$\int_a^b f(g(x))g'(x)\,dx = \int_a^b F'(g(x))g'(x)\,dx = F(g(x))\Big|_a^b = F(g(b)) - F(g(a)).$$

On the other hand, letting $u = g(x)$, we have

$$\int_{g(a)}^{g(b)} f(u)du = F(u)\Big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$

This gives us two options for calculating a definite integral using substitution:

1. We can calculate the antiderivative in terms of x and use the original limits of integration to evaluate the definite integral or
2. we can change the limits of integration when we make the substitution, calculate the antiderivative in terms of u and evaluate using the new limits of integration.

Example Evaluate the following definite integral using both methods

$$\int_0^1 2x\sqrt{x^2+1}\,dx$$

Method 1 In our example above, we calculated $\int 2x\sqrt{x^2+1}\,dx = \frac{2(x^2+1)^{3/2}}{3} + C$. Using the fundamental theorem of calculus, we get

$$\int_0^1 2x\sqrt{x^2+1}\,dx = \frac{2(x^2+1)^{3/2}}{3}\Big|_0^1 = \frac{2(2)^{3/2} - 2(1)^{3/2}}{3} = \frac{4\sqrt{2}-2}{3}.$$

Method 2 As in the example above, we substitute $u = x^2 + 1$. When we change the variable, we also change the limits of integration. When $x = 0$, $u = u(x) = u(0) = 1$, when $x = 1$, $u = u(x) = u(1) = 2$. Our transformed integral is now given by

$$\int_0^1 2x\sqrt{x^2+1}\,dx = \int_1^2 \sqrt{u}\,du = \frac{2u^{3/2}}{3}\Big|_1^2 = \frac{2(2)^{3/2} - 2(1)^{3/2}}{3} = \frac{4\sqrt{2}-2}{3}.$$

Example Evaluate the following definite integrals:

$$\int_0^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx, \qquad \int_2^3 x\sqrt{x^2+1}\,dx.$$

Even and Odd Functions

Sometimes we can use symmetry to make evaluation of integrals easier:
If f is an even function ($f(x) = f(-x)$), then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.
If f is an odd function ($f(x) = -f(-x)$), then $\int_{-a}^{a} f(x)dx = 0$

Example Evaluate the following definite integrals:

$$\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \tan^5 x dx, \qquad \int_{-1}^{1} x^4 + x^2 + 1dx.$$

Extra Examples (Please attempt these before you check the solutions)

Example Find the following indefinite integrals:

$$\int \frac{x}{\sqrt{x^2+1}}\, dx, \qquad \int \sin(2x+1)\, dx$$

Example (tricky - ish) Find the following :

$$\int \sin^2 x \cos^3 x\, dx$$

Example Evaluate the following definite integrals:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^3\theta \cos\theta\, d\theta, \qquad \int_1^2 \frac{x}{\sqrt{x^2+1}}\, dx \quad \text{(Use results from previous example)}$$

Extra Examples Solutions

Example Find the following indefinite integrals:

$$\int \frac{x}{\sqrt{x^2+1}}\,dx, \qquad \int \sin(2x+1)\,dx$$

Ex 1.

$$\int \frac{x}{\sqrt{x^2+1}}\,dx$$

Let $u = x^2 + 1$,
$du = 2xdx \to xdx = \frac{du}{2}$.

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sqrt{u}}\frac{du}{2} = \frac{1}{2}\int \frac{1}{\sqrt{u}}du = \frac{1}{2}\frac{\sqrt{u}}{(1/2)} + C = \sqrt{x^2+1} + C$$

Ex 2.

$$\int \sin(2x+1)\,dx$$

Let $u = 2x + 1$,
$du = 2dx \to dx = \frac{du}{2}$.

$$\int \sin(2x+1)\,dx = \int \sin(u)\,\frac{du}{2} = \frac{1}{2}\int \sin u\,du = \frac{-\cos u}{2} + C$$

Example (tricky - ish) Find the following :

$$\int \sin^2 x \cos^3 x\,dx$$

We let $u = \sin x$ and replace the extra $\cos^2 x$ by $1 - u^2$. We get $du = \cos x\,dx$ and

$$\int \sin^2 x \cos^3 x\,dx = \int u^2(1-u^2)\,du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

Example Evaluate the following definite integrals:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^3\theta\cos\theta\, d\theta, \qquad \int_1^2 \frac{x}{\sqrt{x^2+1}}\, dx \ \ \text{(Use results from previous example)}$$

Ex 1:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^3\theta\cos\theta\, d\theta$$

Let $u = \sin\theta$,
then $du = \cos\theta\, d\theta$.
Changing the limits, we get
$u(\frac{\pi}{4}) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$
$u(\frac{\pi}{3}) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sin\theta)^3\cos\theta\, d\theta = \int_{u(\frac{\pi}{4})}^{u(\frac{\pi}{3})} u^3 du = \int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} u^3 du = \frac{u^4}{4}\Big|_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} = \frac{1}{4}\left[\frac{(\sqrt{3})^4}{16} - \frac{1}{(\sqrt{2})^4}\right]$$

$$= \frac{1}{4}\left[\frac{9}{16} - \frac{1}{4}\right] = \frac{1}{4}\left[\frac{5}{16}\right] = \frac{5}{64}.$$

Ex. 2 (using method 1): Above, we saw that

$$\int \frac{x}{\sqrt{x^2+1}}\, dx = \sqrt{x^2+1} + C$$

So

$$\int_1^2 \frac{x}{\sqrt{x^2+1}}\, dx = \sqrt{x^2+1}\Big|_1^2 = \sqrt{4+1} - \sqrt{1+1} = \sqrt{5} - \sqrt{2}.$$